More complex-valued functions

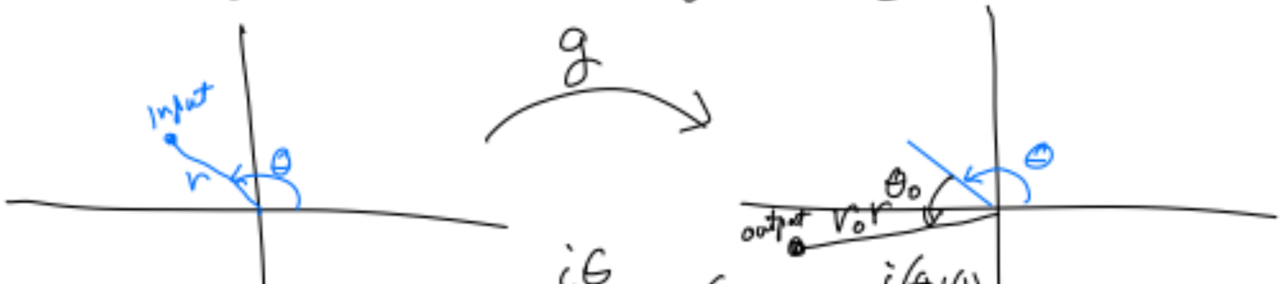
– specifically, those defined by Taylor series – where we have allowed complex values for the variable in the series.

- polynomials

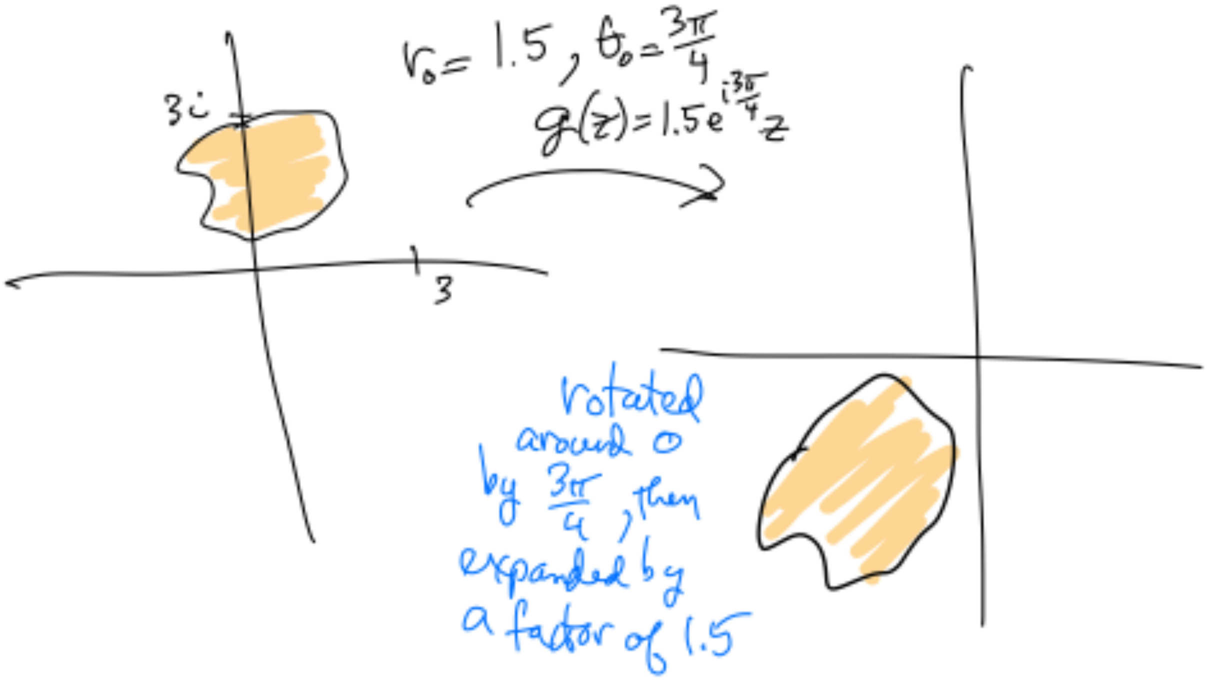
$$f(z) = 3z^2 - 2z + 1$$
$$= 1 - 2z + 3z^2 + 0z^3 + 0z^4 + \cdots$$

We want to imagine complex-valued functions of $\mathbb{C}$ as mapping regions of $\mathbb{C}$ to other regions of $\mathbb{C}$.

Examples ① $g(z) = cz$, where c is a constant $r_0 e^{i\theta_0} = c$



$$z = re^{i\theta} \Rightarrow g(z) = r r_0 e^{i(\theta+\theta_0)}$$



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(2) $f(z) = \underbrace{Cz + V}$

$C = r_0 e^{i\theta_0} \qquad V = x_0 + iy_0$

$f(z)$ is the same a $g(z)$, followed by a translation of $x_0 + iy_0$ (x_0 units right, y_0 units up)

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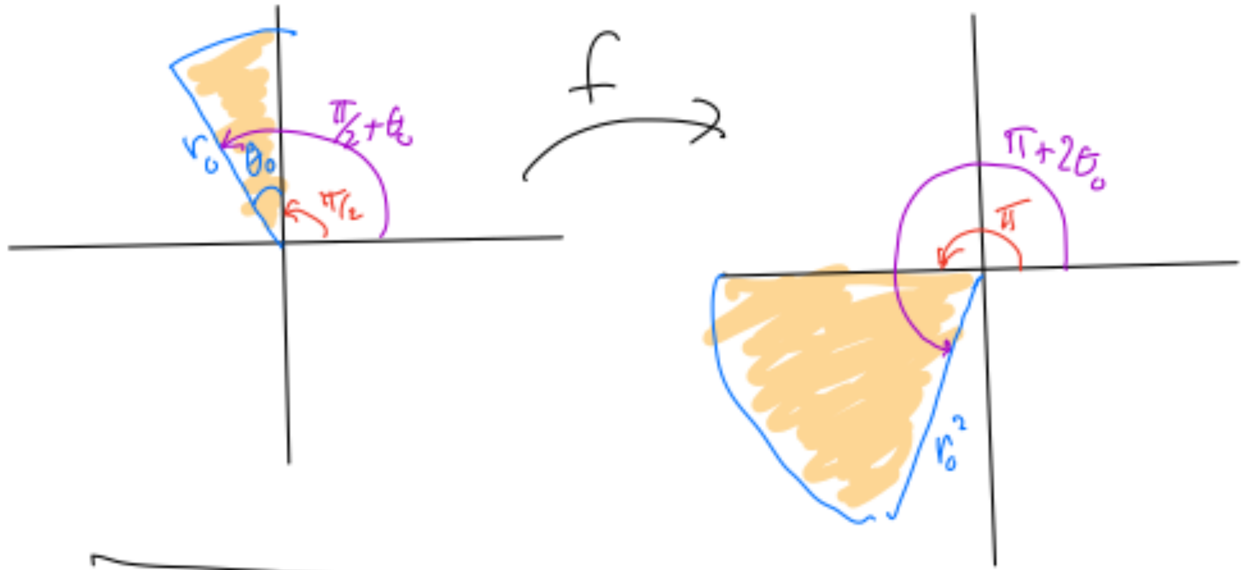
(3) $f(z) = z^2$

If $z = re^{i\theta}$ $\quad f(re^{i\theta}) = r^2 e^{2i\theta}$



- $f(z)=e^z = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$

— Note $e^z e^w = e^{z+w}$

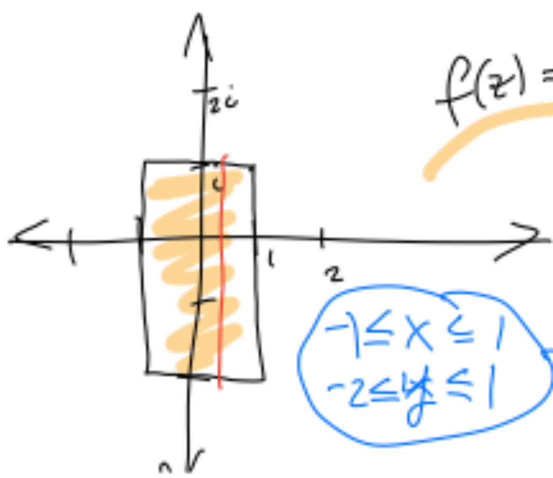
If $z=x+iy$ with $x,y\in\mathbb{R}$, then

$$e^z = e^{x+iy} = e^x\left(e^{iy}\right)$$

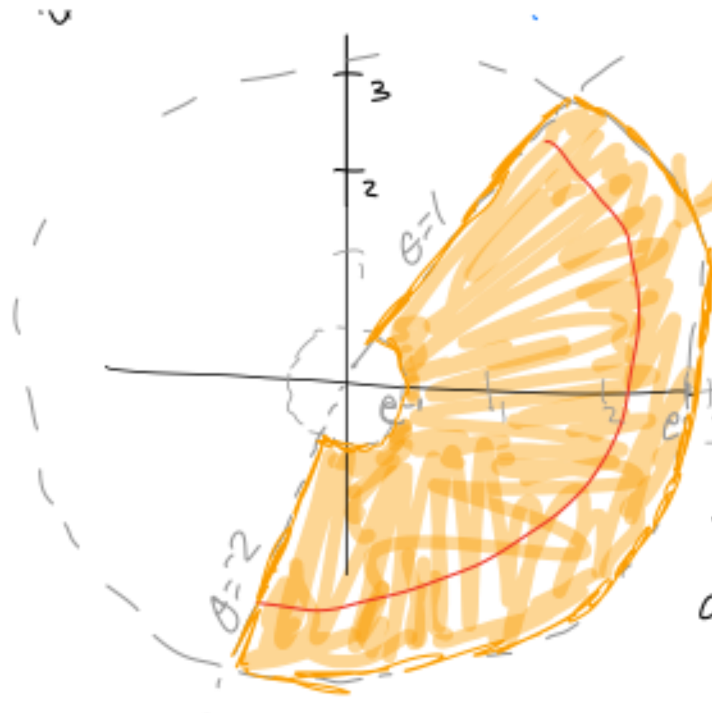
$$= e^x(\cos(y)+i\sin(y))$$

easier to understand in polar.

$f(z)=e^z$



$$\begin{matrix} -1\le x\le 1 \\ -2\le y\le 1 \end{matrix} \xrightarrow{f} \begin{matrix} e^{-1}\le r\le e^{1} \\ -2\le \theta\le 1 \end{matrix}$$



The image is a portion of an annulus

More functions:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

We can derive the properties from the exponential properties.

Do the trig identities work?

Let's check $\sin^2(z) + \cos^2(z) = 1$:

$$LHS = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2$$

$$= \frac{\cancel{e^{2iz}} - 2 + \cancel{e^{-2iz}}}{-4} + \frac{\cancel{e^{2iz}} + 2 + \cancel{e^{-2iz}}}{4}$$

$$= \frac{1}{2} + \frac{1}{2} = \boxed{1}.$$

(Actually: the secret — all the trig identities work with complex #s plugged in!)

[One possible proof: can derive all of them from the exponential properties (which can be derived from the Taylor series of e^z)]

To look at $\sin(z)$ & $\cos(z)$.

If $z = x+iy$, $x, y \in \mathbb{R}$

$\Rightarrow \sin(z) = \sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy)$

$$\cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{-y}+e^{y}}{2} = \cosh(y).$$

$$\sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y}-e^{y}}{2i} = i\frac{(e^{y}-e^{-y})}{2}$$

$$= i\sinh(y).$$

$$\Rightarrow \sin(z) = \underbrace{\sin(x)\cosh(y)}_{\text{real part}} + i\underbrace{\cos(x)\sinh(y)}_{\text{imag part}}.$$

Note: $\sin(z)$ & $\cos(z)$ are not bounded in $\mathbb{C}$.

If we fix x, vary y.

$$\cosh(y) \to \infty \text{ as } y \to \pm\infty$$

$$\sinh(y) \to \infty \text{ as } y \to \infty$$

$$\sinh(y) \to -\infty \text{ as } y \to -\infty.$$

But they are bounded on $\mathbb{R}$

$$\left.\begin{aligned} -1 \le \sin(z) \le 1 \\ -1 \le \cos(z) \le 1 \end{aligned}\right\} \text{ if } z \in \mathbb{R} \subseteq \mathbb{C}.$$

$$\left.\begin{aligned} \mathrm{Log}(z) \\ \log(z) \end{aligned}\right\} \text{ be careful of branches of } \arg(z)$$

$$\underset{\text{multivalued log}}{\log(z)} = \mathrm{Log}(|z|) + i \underset{\text{multivalued angle}}{\arg(z)}$$

$$\mathrm{Log}(z) = \mathrm{Log}(|z|) + i\, \underset{\text{principal arg}(z)}{\mathrm{Arg}(z)} \quad -\pi < \mathrm{Arg}(z) \le \pi$$

Complex derivatives.

Definition: Let $f: U \to \mathbb{C}$, where $U \subseteq \mathbb{C}$. The complex derivative of f at $a \in U$ is defined to be

$$f'(a) = \frac{df}{dz}(a) = \underbrace{\lim_{\cancel{z \to a}\; h \to 0}}_{\text{complex limits}} \frac{f(a+h) - f(a)}{h}.$$

(Also $f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$.)

This may or may not exist.

Often, one has to assume that a is an interior point of U.

Because of the algebraic limit theorems all the derivative formulas you learned in calculus still work, but they mean something much more.

It's actually pretty hard for $f'(z)$ to exist.

$\boxed{\text{Example}}$ $f(z) = \mathrm{Re}(z)$

$f(x+iy) = x \leftarrow$ This a C^∞ fn. (all of its multiple partial derivs exist and are continuous.)

But.

$$\frac{f(a+h) - f(a)}{h} \qquad a = 0$$

$$\frac{f(h) - f(0)}{h} = \frac{\mathrm{Re}(h)}{h}$$

$\lim\limits_{h\to 0}$ along x-axis.

$h = \alpha(t) = t \in \mathbb{C}$. $\qquad \lim\limits_{t\to 0} \alpha(t) = 0$

$$\lim_{t\to 0} \frac{\mathrm{Re}(\alpha(t))}{\alpha(t)} = \lim_{t\to 0} \frac{t}{t} = \boxed{1}$$

$\beta(t) = it \in \mathbb{C}$. $\lim_{t\to 0} \beta(t) = 0$

$$\lim_{t\to 0} \frac{\mathrm{Re}(\beta(t))}{\beta(t)} = \lim_{t\to 0} \frac{0}{it} = \boxed{0} \neq 1$$

$\therefore \lim_{h\to 0} \frac{\mathrm{Re}(h)}{h}$ does not exist.

$$\boxed{F(z) = \mathrm{Re}(z) \text{ is not complex diff'ble at } z=0!}$$

(in fact, the derivative does not exist anywhere!)